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# A Polynomial Approach to Topological Analysis, II

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## 1. INTRODUCTION

In 1957 Weiner and Porcelli [3] gave a derivation based on the fundamental theorem of algebra of the Cauchy inequality for polynomials which states that if $P(z) = \sum_0^n a_k z^k$ is a polynomial such that $|P(z)| \leqslant 1$ for all $z$ with $|z| = 1$, then $|a_k| \leqslant 1$ for $k = 0, 1, 2,..., n$. In 1961 Porcelli and Connell [2] gave a greatly simplified derivation of the same. In this paper we show how this result coupled with the Stone–Weierstrass theorem can be exploited to give a development of complex variable theory for *once* continuously differentiable functions. Our discussion is aimed at a derivation of the Laurent expansion for functions defined on an annular region. A byproduct of our development is the construction of a functional acting on the space $C(B)$ of complex-valued, continuous functions on the circle $|z| = 1$ which may be interpreted as an integral.

We shall use the maximum modulus theorem for polynomials (which can be proved in an elementary way). This theorem suffices to give us, in Theorem 3.1., the Cauchy inequality for polynomials. This, in turn, trivially implies the Cauchy inequality for functions of the form $\sum_{-n}^n a_k z^k$.

We denote by $T$ the set of all these functions restricted to $B$. Let $f \in C(B)$. From the Stone–Weierstrass theorem, $f$ is the limit of a sequence of elements of $T$. In Theorem 3.3 the Cauchy inequality is used to define from this approximating sequence a formal series $\sum_{-\infty}^{\infty} a_k(f)\, z^k$ such that if $f(z) = \sum_{-n}^n b_k z^k$ throughout $|z| = 1$, then $a_k(f) = b_k$ for $k = -n,..., n$. It then turns out that if $f$ is continuously differentiable in an annular region $\alpha < |z| < \beta$, where $0 \leqslant \alpha < 1 < \beta$, then we have $f(z) = \sum_{-\infty}^{\infty} a_k(f)\, z^k$. To facilitate the argument, we set $L(f) = a_0(f)$ for $f \in C(B)$, and separately develop the properties of $L$, thus obtaining an integral. For $f \in C(B)$, $p$ an integer, $a_p(f)$ coincides with the $p$-th Fourier coefficient of $f$.

## 2. Notation

Let $K$ denote the complex plane. For $\delta > 0$, set $U(\delta) = \{z \in K; |z| < \delta\}$, $B(\delta) = \{z \in K; |z| = \delta\}$, $U = U(1)$ and $B = B(1)$. For any continuous function $f$ defined on a set $S$ containing $B$, set $\|f\| = \max\{|f(z)|; z \in B\}$.

## 3. The Basic Coefficients

THEOREM 3.1 (Cauchy inequality). *If a polynomial* $P(z) = \sum_0^n a_k z^k$ *satisfies* $\|P\| \leqslant 1$, *then* $|a_k| \leqslant 1$ *for* $k = 0, 1,..., n$.

*Proof.* This proof is due to Porcelli and Connell [2]. Trivially, the theorem holds for polynomials of degree zero. Suppose it holds for polynomials of degree $n$ or less, and let $P(z) = \sum_0^{n+1} a_k z^k$ be a polynomial of degree $n + 1$ such that $\|P\| \leqslant 1$.

Let $\theta \in B$, and set $Q(z) = 2^{-1}[P(z) - P(\theta z)]$. Then $Q(0) = 0$. Let the polynomial $Q_0(z)$ be defined by $Q(z) = zQ_0(z)$. Then $\|Q_0\| = \|Q\| \leqslant 1$. By the induction hypothesis, $|2^{-1}a_k(1 - \theta^k)| \leqslant 1$ for $k = 1,..., n + 1$. Taking $\theta$ such that $\theta^k = -1$, we have $|a_k| \leqslant 1$ for $k = 1, 2,..., n + 1$. Finally, by the maximum modulus theorem, $|a_0| = |P(0)| \leqslant \|P\| \leqslant 1$.

COROLLARY 3.2. *If* $P(z) = \sum_{-n}^{n} a_k z^k$ *is such that* $\|P\| \leqslant 1$, *then* $|a_k| \leqslant 1$ *for all* $k$.

*Proof.* Set $Q(z) = z^n P(z)$. *Then* $\|Q\| = \|P\| \leqslant 1$, and, by Theorem 3.1, all $|a_k|$ are $\leqslant 1$.

Let $T_0$ be the family of all real valued elements of $T$. Clearly, $T$ and $T_0$ are subalgebras of $C(B)$, containing the constant function 1. Let $z_1, z_2 \in B$, $z_1 \neq z_2$, and set $P(z) = |z - z_1|^2 = (z - z_1)(\bar{z} - \bar{z}_1) = -z_1 z^{-1} + 2 - \bar{z}_1 z$ for $z \in B$. Then $P \in T_0$ and $P(z_2) \neq P(z_1)$. Thus, $T_0$ separates points of $B$ and, hence, from the Stone–Weierstrass theorem, the closure of $T_0$ in $C(B)$ is the space $C_0(B)$ of all real-valued elements of $C(B)$. Hence, the closure of $T$ is $C(B)$.

THEOREM 3.3. *Let* $f \in C(B)$. *Then there exists a sequence* $\{a_k(f)\}_{k=-\infty}^{\infty}$ *such that* $|a_k(f)| \leqslant \|f\|$ *for all* $k$, *and such that if* $P(z) = \sum_{-n}^{n} b_k z^k$ *satisfies* $\|P - f\| \leqslant \delta$, *then* $|a_k(f) - b_k| \leqslant \delta$ *for* $k = 0, \pm 1,..., \pm n$.

*Proof.* For $n = 1, 2,...$, let $P_n(z) = \sum_{-\infty}^{\infty} a_{nk} z^k$ (with $a_{nk} = 0$ for all $k$ with $|k| \geqslant$ some $k_0(n)$) be such that $\|P_n - f\| \leqslant 1/n$. Then for $n, m = 1, 2,...$ and $z \in B$,

$$\left| \sum (a_{nk} - a_{mk}) z^k \right| = |P_n(z) - P_m(z)|$$
$$\leqslant \|P_n - P_m\| \leqslant \|P_n - f\| + \|f - P_m\| \leqslant n^{-1} + m^{-1},$$

and, hence, from Corollary 3.2, $| a_{nk} - a_{mk} | \leqslant n^{-1} + m^{-1}$ for $k = 0, \pm 1, \pm 2, \ldots$ . Thus, for each $k$, $a_{1k}, a_{2k}, \ldots$ converges to a limit $a_k(f)$ and $| a_k(f) - a_{nk} | \leqslant 1/n$ for $n = 1, 2, \ldots$ . Let $z \in B$. Then, for $n = 1, 2, \ldots$,

$$\left| \sum a_{nk} z^k \right| = | P_n(z) | \leqslant \| P_n \| \leqslant \| P_n - f \| + \| f \| \leqslant n^{-1} + \| f \|,$$

and, hence, from Corollary 3.2, $| a_{nk} | \leqslant \| f \| + n^{-1}$, for all $k$. Letting $n \to \infty$, we obtain $| a_k(f) | \leqslant \| f \|$.

For $z \in B$, setting $b_k = 0$ for $| k | > n$, we have

$$\left| \sum (a_{nk} - b_k) z^k \right| = | P(z) - P_n(z) |$$
$$\leqslant \| P - P_n \| \leqslant \| P - f \| + \| f - P_n \| \leqslant \delta + n^{-1}.$$

Hence $| a_{nk} - b_k | \leqslant \delta + n^{-1}$. Letting $n \to \infty$, we obtain

$$| a_k(f) - b_k | \leqslant \delta.$$

## 4. The Functional $a_0(f)$

For $f \in C(B)$, set $L(f) = a_0(f)$.

Theorem 4.1. *$L$ is a bounded linear functional on $C(B)$ such that, for $f \in C(B)$, $\theta \in B$,*

(1) $L(1) = 1$.

(2) $\| L \| = 1$.

(3) $L(f_\theta) = L(f)$, *where* $f_\theta(z) = f(\theta z)$.

(4) $L(\bar{f}) = \overline{L(f)}$ *and, hence, if $f$ is real-valued, $L(f)$ is real.*

(5) *If $f(z) \geqslant 0$ for all $z \in B$, then $L(f) \geqslant 0$.*

(6) *If $f(z) \geqslant 0$ for all $z \in B$ and $f(z_0) \neq 0$ for some $z_0 \in B$, then $L(f) > 0$.*

*Proof.* From Theorem 3.4, $| L(f) | = | a_0(f) | \leqslant \| f \|$ and, thus, $\| L \| \leqslant 1$. Trivially, $L(1) = a_0(1) = 1$ and, thus, $| L(1) | = 1$ and $\| L \| = 1$.

We now show that $L$ is linear. Let $f, g \in C(B)$ and let $\delta > 0$. Then there exist finite series $P(z) = \sum a_k z^k$ and $Q(z) = \sum b_k z^k$ such that $\| P - f \| < \delta/4$, $\| Q - g \| < \delta/4$. Whence,

$$\| (f + g) - (P + Q) \| \leqslant \| f - P \| + \| g - Q \| < (\delta/4) + (\delta/4) = \delta/2$$

and, hence, from Theorem 3.3, $| a_0 - a_0(f)| \leqslant \delta/4$, $| b_0 - a_0(g)| \leqslant \delta/4$, and $|(a_0 + b_0) - a_0(f + g)| \leqslant \delta/2$. Thus,

$$\begin{aligned} | a_0(f + g) - a_0(f) - a_0(g)| &\leqslant | a_0(f + g) - a_0 - b_0 | \\ &\quad + | a_0 - a_0(f)| + | b_0 - a_0(g)| \\ &\leqslant (\delta/4) + (\delta/4) + (\delta/2) = \delta. \end{aligned}$$

Since $\delta$ is arbitrary, $L(f + g) = a_0(f + g) = a_0(f) + a_0(g) = L(f) + L(g)$. Similarly, for $h \in C(B)$ and $c$ a constant, $L(ch) = cL(h)$.

We now establish the rotation invariance property (3). For $z \in B$, $|f(\theta z) - P(\theta z)| < \delta/4$ and, from Theorem 3.3, $| a_0(f_\theta) - a_0 | \leqslant \delta/4$. Thus

$$| a_0(f_\theta) - a_0(f)| \leqslant | a_0(f_\theta) - a_0 | + | a_0 - a_0(f)| \leqslant (\delta/4) + (\delta/4) = \delta/2.$$

Since $\delta$ is arbitrary, $L(f_\theta) = a_0(f_\theta) = a_0(f) = L(f)$.

We now establish (4). For $z \in B$,

$$\left|\overline{f(z)} - \sum \overline{a_{-k}} z^k\right| = \left|\overline{f(z)} - \sum \bar{a}_k z^{-k}\right| = \left|f(z) - \sum a_k z^k\right| < \delta/4$$

and, hence, from Theorem 3.3, $| a_0(\bar{f}) - \overline{a_0} | \leqslant \delta/4$. Thus

$$| a_0(\bar{f}) - \overline{a_0(f)}| \leqslant | a_0(\bar{f}) - \overline{a_0} | + | \overline{a_0} - \overline{a_0(f)}| \leqslant (\delta/4) + (\delta/4) = \delta/2.$$

Since $\delta$ is arbitrary, we have $L(\bar{f}) = a_0(\bar{f}) = \overline{a_0(f)} = \overline{L(f)}$.

We establish (5). Without loss of generality, suppose $0 \leqslant f(z) \leqslant 1$ for $z \in B$. Then $1 - L(f) = L(1) - L(f) = L(1 - f) \leqslant | L(1 - f)| \leqslant \| 1 - f\| \leqslant 1$, and, thus, $L(f) \geqslant 0$.

We now prove (6). Without loss of generality, suppose $f(1) > 0$. Then there exist a $\delta > 0$ such that $f(z) > 0$ for all $z \in B$ satisfying $| z - 1 | < \delta$. Now there exist $\theta_1, \theta_2, \ldots, \theta_n \in B$ such that given a $z \in B$, there exists a $\theta_k$ with $| \theta_k z - 1 | < \delta$, so that $f_{\theta_k}(z) = f(\theta_k z) > 0$. Setting $f_0 = \sum_1^n f_{\theta_k}$, we obtain $f_0(z) > 0$ for all $z \in B$. With $\rho = \min\{f_0(z); z \in B\}$, we have $L(f_0) \geqslant L(\rho) = \rho > 0$.

THEOREM 4.2. *For $f \in C(B)$ and $p$ an integer, $a_p(f) = L(z^{-p}f)$.*

*Proof.* Let $\delta > 0$. Then there exists a finite series $P(z) = \sum b_k z^k$ such that $\| P - f\| < \delta/2$. Now, for $z \in B$,

$$\left| z^{-p}f(z) - \sum b_{k+p} z^k \right| = | z^{-p}[f(z) - P(z)]| = |f(z) - P(z)| < \delta/2,$$

and, hence, from Theorem 3.3, $|b_p - a_0(z^{-p}f)| \leqslant \delta/2$. Thus,

$$|a_p(f) - a_0(z^{-p}f)| \leqslant |a_p(f) - b_p| + |b_p - a_0(z^{-p}f)| \leqslant (\delta/2) + (\delta/2) = \delta.$$

Since $\delta$ is arbitrary, $a_p(f) = L(z^{-p}f)$.

THEOREM 4.3. *Let $f \in C(B)$ be such that $a_k(f) = 0$ for all $k$. Then $f(z) = 0$ for all $z \in B$.*

*Proof.* From Theorem 4.2, $L(Qf) = 0$ for all $Q$ in $T$. Set $M = \max\{1, \|f\|\}$ and let $\delta > 0$. There exists a $P \in T$ such that $\|P - f\| < \delta/M$. Then

$$\| |f|^2 - \bar{P}f\| = \|f(\bar{f} - \bar{P})\| \leqslant \|f\| \cdot \|\bar{f} - \bar{P}\| \leqslant M\|f - P\| < \delta$$

and, hence,

$$|L(|f|^2)| = |L(|f|^2) - L(\bar{P}f)| = |L(|f|^2 - \bar{P}f)| \leqslant \| |f|^2 - \bar{P}f\| < \delta.$$

Since $\delta$ is arbitrary, $L(|f|^2) = 0$. From (6) of Theorem 4.1, $f(z) = 0$ for all $z \in B$.

*Remark.* Let $f \in C(B)$. Then (see Theorem 5.2)

$$L(f) = (2\pi i)^{-1} \int_B f(z)\, z^{-1}\, dz = (2\pi)^{-1} \int_0^{2\pi} f(e^{i\theta})\, d\theta.$$

Thus, by Theorem 4.2, $a_p(f)$ ($p = 0, \pm 1, \pm 2, \ldots$) is the $p$-th Fourier coefficient $(2\pi)^{-1} \int_0^{2\pi} f(e^{i\theta})\, e^{-ip\theta}\, d\theta$ of $f$. No result of this paper requires knowledge of these interpretations.

## 5. THE LAURENT EXPANSION

Let $0 \leqslant \alpha < 1 < \beta$ and let $g$ be a complex, continuous function on the annulus $\alpha < |z| < \beta$. For $p = 0, \pm 1, \pm 2, \ldots$, $\alpha < r < \beta$, we set $a_p(r, g) = a_p(g_r)\, r^{-p}$, where $g_r(z) = g(rz)$ for all $z \in B$.

THEOREM 5.1. *Let $0 \leqslant \alpha < 1 < \beta$ and let $f$ be a complex, continuously differentiable function on the annulus $\alpha < |z| < \beta$. Then for $p = 0, \pm 1, \pm 2, \ldots$, and $\alpha < \rho < \beta$, we have $a_p(\rho, f) = a_p(1, f) = a_p(f)$.*

*Proof.* It suffices to show that $a_0(\rho, f) = a_0(f)$, since from Theorem 4.2 we would have, for every integer $p$,

$$a_p(\rho, f) = a_p(f_\rho)\, \rho^{-p} = a_0(z^{-p}f_\rho)\, \rho^{-p} = a_0[(z^{-p}f)_\rho] = a_0(z^{-p}f) = a_p(f).$$

Let $\alpha < \alpha_0 < 1 < \beta_0 < \beta$ and let $\epsilon > 0$. Then there exists a $\delta > 0$ such that if $\alpha_0 \leqslant |x| \leqslant \beta_0$, $\alpha_0 \leqslant |y| \leqslant \beta_0$ and $|y - x| < \delta$, then

$$|[f(y) - f(x)](y - x)^{-1} - f'(x)| < \epsilon/2.$$

Let $\alpha_0 \leqslant r < s \leqslant \beta_0$, $|s - r| < \delta$, and let $|\theta| = 1$, $0 < |\theta - 1| < \delta/\beta$. Let $z$ be a point of $B$. Then $|sz - rz| = |s - r| < \delta$, $|rz - r\theta z| = r|1 - \theta| \leqslant \beta |1 - \theta| < \delta$,

$$\begin{aligned}(1)\quad & |[f_s(z) - f_r(z)](s - r)^{-1} - f'(rz)z| \\ &= |[f(sz) - f(rz)](sz - rz)^{-1} - f'(rz)| < \epsilon/2,\end{aligned}$$

and

$$(2)\quad |[f_{r\theta}(z) - f_r(z)](r\theta - r)^{-1} - f'(rz)z| < \epsilon/2.$$

Adding, we obtain

$$|[f_s(z) - f_r(z)](s - r)^{-1} - [f_{r\theta}(z) - f_r(z)](r\theta - r)^{-1}| < \epsilon,$$

and thus

$$(3)\quad |L[(f_s - f_r)(s - r)^{-1} - (f_{r\theta} - f_r)(r\theta - r)^{-1}]| \leqslant \epsilon.$$

Now, since $|\theta| = 1$, $L(f_{r\theta}) = L(f_r)$, and, thus, from (3),

$$|L[(f_s - f_r)(s - r)^{-1}]| \leqslant \epsilon.$$

Consequently,

$$|a_0(s, f) - a_0(r, f)| = |a_0(f_s) - a_0(f_r)| = |a_0(f_s - f_r)| \leqslant \epsilon |s - r|.$$

We may assume $\rho \neq 1$. Subdividing suitably the interval with endpoints $\rho$ and 1, we obtain $|a_0(\rho, f) - a_0(f)| \leqslant \epsilon |1 - \rho|$. Since $\epsilon$ is arbitrary, $a_0(\rho, f) = a_0(f)$.

THEOREM 5.2 (Laurent expansion). *Let $0 \leqslant \alpha < 1 < \beta$ and let $f$ be a complex, continuously differentiable function on $S = \{z; \alpha < |z| < \beta\}$. Then the series $\sum a_k(f) z^k$ converges uniformly to $f$ on compact subsets of $S$.*

*Proof.* Let $C$ be a compact subset of $S$. Choose $r$, $s$ and $\rho$ such that $0 < \rho < 1$, $\alpha < r < s < \beta$ and such that $|z|\rho \geqslant r$, $|z|/\rho \leqslant s$ for all $z$ in $C$. Set $M = \max\{|f(t)|; r \leqslant |t| \leqslant s\}$ and let $p$ be an integer. From Theorem 3.3, $|a_p(f_r)| \leqslant M$, and, hence, applying Theorem 5.1,

$$|a_p(f)| = |a_p(r, f)| = |a_p(f_r)| r^{-p} \leqslant Mr^{-p}.$$

Similarly, $|a_p(f)| \leqslant Ms^{-p}$. Thus, if $z$ belongs to $C$ then,

$$|a_p(f)\, z^p| \leqslant M\rho^p \quad (p = 0, 1,...), \qquad |a_{-p}(f)\, z^{-p}| \leqslant M\rho^p \quad (p = 1, 2,...),$$

and so, the series $g(z) = \sum a_k(f)\, z^k$ is absolute and uniformly convergent on $C$.

Let $z \in S$ and set $r_0 = |z|$. Then, for any integers $p$, $k$, $k \neq p$, we have $a_p(z^k) = 0$. Therefore, for $p = 0, \pm 1, \pm 2,...$,

$$a_p(g_{r_0}) = a_p\left(\sum a_k(f)\, r_0{}^k z^k\right) = \sum a_k(f)\, r_0{}^k a_p(z^k) = a_p(f)\, r_0{}^p = a_p(f_{r_0}).$$

Thus, for all integers $p$, $a_p(g_{r_0} - f_{r_0}) = 0$, and, hence, fromTheorem 4.3, $g(r_0 x) - f(r_0 x) = 0$ for all $x \in B$. In particular, since $z = r_0 x$ for some $x \in B$, $g(z) = f(z)$.

COROLLARY 5.3 (Removable Singularity Theorem). *Let $f$ be a complex function, continuously differentiable on $0 < |z| < R$ and continuous at 0. Then $f$ is continuously differentiable at 0.*

*Proof.* Without loss of generality, we can assume $R > 1$. From Theorem 5.2, the series $\sum a_k(f)\, z^k$ converges to $f$ on $0 < |z| < R$. Moreover, for every positive integer $p$ and every $\rho$, $0 < \rho \leqslant 1$, we have

$$|a_{-p}(f)| \leqslant \rho^p \max\{|f(z)|; |z| \leqslant 1\};$$

hence $a_{-p} = 0$. Thus, throughout $|z| < R$, $f(z) = \sum_0^\infty a_k(f) z^k$, and the result follows.

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